

Directed polymer in random environment and two points state space

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Abstract:

We give an exact expression for the partition function of a continuous time DPPE on a two points state space.

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1. Introduction

We are aware of just three¹ models of directed polymer in a random environment for which the free energy at finite inverse temperature is well known:

1. The discrete polymer in dimension $d = 1$ with simple random walk paths and a log-gamma environment and boundary conditions: see [Seppäläinen \(2012\)](#).
2. The continuous time directed polymer with Poisson process paths and a Brownian environment : see [O'Connell and Yor \(2001\)](#).
3. The continuum random polymer with brownian paths and a white noise environment : see [Amir, Corwin and Quastel \(2011\)](#).

The origin of this note is to find the simplest possible model where some direct computations can be performed. Of course, this model is much less interesting than the three models above. Note also that we are even unable to generalize it to a three points state space !

Let us consider the simplest model of continuous time directed polymer in a random environment. Let $\omega = (\omega(t))_{t \geq 0}$ be the continuous time Markov chain on $\{1, 2\}$ with generator:

$$Lf(1) = f(2) - f(1), \quad Lf(2) = f(1) - f(2), \quad (1.1)$$

that is the chain that spends an exponential time on 1 (resp 2) and the jumps on 2 (resp. 1). We let P_i denote the law of the Markov chain starting from i , E_i the associated expectation, and (W, \mathcal{W}) the path space of piecewise constant cadlag functions from $[0, +\infty[$ to $\{1, 2\}$. We set $P = P_1$ and $E = E_1$.

The random environment consists of two independent standard Brownian motions $(B_i(t), t \geq 0)$ defined on another probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For any $t > 0$ the (random) *polymer measure* μ_t is the probability defined on the path space (W, \mathcal{W}) by

$$\mu_t(d\omega) = \frac{1}{Z_t} e^{\beta H_t(\omega) - t\beta^2/2} \mathbb{P}(d\omega)$$

where $\beta \geq 0$ is the inverse temperature, the Hamiltonian is

$$H_t(\omega) = \int_0^t dB_{\omega(s)}(s)$$

¹If some interested reader finds another example where an exact computation of the free energy occurs, we are more than willing to incorporate it in this list

and the partition function is

$$Z_t = Z_t(\beta) = E \left[e^{\beta H_t(\omega) - t\beta^2/2} \right],$$

It is well known, and for example established in (Carmona and Hu, 2006, Proposition 2.4) that the point to point partition function

$$Z_t(x, y) := E_x \left[e^{\beta H_t(\omega) - t\beta^2/2} \mathbf{1}_{(\omega(t)=y)} \right] \quad (1.2)$$

satisfy the Discrete Stochastic Heat Equation:

$$dZ_t(x, y) = LZ_t(x, \cdot)(y) dt + \beta Z_t(x, y) dB_y(t), \quad (1.3)$$

in the Itô sense with initial conditions $Z_0(x, y) = \delta_x(y)$ (from now on we shall fix as a starting point $x = 1$). Furthermore, the free energy is well defined

$$p(\beta) := \lim_{t \rightarrow +\infty} \frac{1}{t} \log Z_t(\beta) \quad (\text{a.s. and in } L^1(\mathbb{P})), \quad (1.4)$$

and is given by the limit (see (Carmona and Hu, 2006, Formula (15)))

$$p(\beta) = -\frac{\beta^2}{2} \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbb{E} [I_s] ds, \quad (1.5)$$

with I_t the overlap

$$I_t = \mu_t^{\otimes 2}(\omega_1(t) = \omega_2(t)) = \frac{1}{Z_t^2} \sum_{y=1}^2 Z_t(1, y)^2.$$

Theorem 1.1. *For this model of DPRE the mean overlap converges:*

$$\lim_{t \rightarrow +\infty} \mathbb{E} [I_t] = \alpha_-(\beta) \quad (1.6)$$

with $\alpha_-(\beta) \in (0, 1)$ the smallest root of the polynomial

$$P_\beta(X) = 3\beta^2 X^2 - (5\beta^2 + 4)X + 2(1 + \beta^2).$$

Consequently the free energy is

$$p(\beta) = -\frac{\beta^2}{2} \alpha_-(\beta).$$

Observe that as $\beta \rightarrow 0^+$, $\alpha_-(\beta) \rightarrow \frac{1}{2}$ as expected.

2. Proof of Theorem 1.1

To simplify notations we let $X_i(t) = Z_t(1, i)$ and set $Z_0 = 1$ so that we have $X_1(0) = 1, X_2(0) = 0, I_0 = 1$ and (X_1, X_2) is solution of the following simple system of stochastic differential equations:

$$\begin{cases} dX_1(t) = (X_2 - X_1) dt + \beta X_1 dB_1(t) \\ dX_2(t) = -(X_2 - X_1) dt + \beta X_2 dB_2(t) \end{cases} \quad (2.1)$$

It is easy to check that $Z_t = X_1(t) + X_2(t)$ is a martingale

$$dZ_t = \beta(X_1 dB_1(t) + X_2 dB_2(t))$$

with quadratic variation

$$d\langle Z \rangle_t = \beta^2(X_1^2 + X_2^2) dt = \beta^2 N_t dt = \beta^2 Z_t^2 I_t dt,$$

where we have set

$$N_t = X_1(t)^2 + X_2(t)^2 \quad \text{so that} \quad I_t = \frac{N_t}{Z_t^2}.$$

Without having to read [Carmona and Hu \(2006\)](#), one can infer directly that

$$\log Z_t = \int_0^t \frac{dZ_s}{Z_s} - \frac{1}{2} \int_0^t \frac{d\langle Z \rangle_s}{Z_s^2} = \int_0^t \frac{dZ_s}{Z_s} - \frac{\beta^2}{2} \int_0^t I_s ds$$

so that

$$\frac{1}{t} \mathbb{E} [\log Z_t] = -\frac{\beta^2}{2} \frac{1}{t} \int_0^t \mathbb{E} [I_s] ds.$$

Let us do now some straightforward computations using Ito's formula

$$\begin{aligned} dX_1^2(t) &= ((\beta^2 - 2)X_1^2 + 2X_1X_2)dt + 2\beta X_1^2 dB_1 \\ dN_t &= (4X_1X_2 + (\beta^2 - 2)N_t)dt + 2\beta(X_1^2 dB_1 + X_2^2 dB_2) = (2Z_t^2 + (\beta^2 - 4)N_t)dt + 2\beta(X_1^2 dB_1 + X_2^2 dB_2) \\ d\langle N, Z \rangle_t &= 2\beta^2(X_1^3 + X_2^3)dt = \beta^2(3Z_t N_t - Z_t^3)dt = \beta^2 Z_t^3(3I_t - 1)dt. \end{aligned}$$

In the last equation, we use the identity $2(a^3 + b^3) = 3(a+b)(a^2 + b^2) - (a+b)^3$. It is the only place where we really use the fact that the state space has only two points.

Let use the notation $U_t \sim V_t$ if $U_t - V_t$ is a martingale. We have:

$$\begin{aligned} dI_t &= \frac{dN_t}{Z_t^2} - \frac{2N_t dZ_t}{Z_t^3} - 2 \frac{d\langle N, Z \rangle_t}{Z_t^3} + 3N_t \frac{d\langle Z \rangle_t}{Z_t^4} \\ &\sim ((\beta^2 - 4)I_t + 2 - 2\beta^2(3I_t - 1) + 3\beta^2 I_t^2)dt \\ &\sim (3\beta^2 I_t^2 - (5\beta^2 + 4)I_t + 2(1 + \beta^2))dt = P(I_t)dt \end{aligned} \tag{2.2}$$

with $a = \frac{5\beta^2 + 4}{6\beta^2} = \frac{5}{6} + \frac{2}{3\beta^2}$ and

$$P(X) = 3\beta^2(X^2 - 2aX + a - \frac{1}{6}) = 3\beta^2(X - \alpha_+)(X - \alpha_-)$$

with $\alpha_{\pm} := a \pm \sqrt{a^2 - a + \frac{1}{6}}$. Since $P(1) = 3\beta^2(1 - a - \frac{1}{6}) = -2$ we have $\alpha_- < 1 < \alpha_+$.

We now take expectations in (2.2) and get

$$\frac{d\mathbb{E} [I_t]}{dt} = \mathbb{E} [P(I_t)]. \tag{2.3}$$

Since $I_0 = 1$, and $P(1) < 0$, at least on a non empty interval $[0, \delta[$, the function $u(t) := \mathbb{E} [I_t]$ is non increasing (something we expected since $0 \leq I_t \leq 1$ donc $0 \leq u(t) \leq 1$).

Assume that there exists $t_0 > 0$ such that $u(t_0) < \alpha_-$. Since $\mathbb{E} [I_t^2] \geq \mathbb{E} [I_t]^2$ we have

$$u'(t) = \mathbb{E} [P(I_t)] \geq P(u(t)).$$

Let $T = \sup \{t < t_0 : u(t) = \alpha_-\}$. On $]T, t_0[$ we have $u'(t) \geq P(u(t)) > 0$ so u strictly increases on $]T, t_0[$, which is absurd since $u(T) = \alpha_-$ and $u(t_0) < \alpha_-$.

Therefore, we have:

$$\forall t > 0, u(t) \geq \alpha_-.$$

Observe that:

$$\begin{aligned} u'(t) &= \mathbb{E} [P(I_t)] = \mathbb{E} [P(I_t) - P(\alpha_-)] = \mathbb{E} [3\beta^2(I_t^2 - \alpha_-^2) - (5\beta^2 + 4)(I_t - \alpha_-)] \\ &= 3\beta^2 \mathbb{E} [(I_t + \alpha_-)(I_t - \alpha_-)] - (5\beta^2 + 4) \mathbb{E} [I_t - \alpha_-] \\ &\leq (3\beta^2(1 + \alpha_-) - (5\beta^2 + 4)) \mathbb{E} [I_t - \alpha_-] = -\lambda \mathbb{E} [I_t - \alpha_-]. \end{aligned}$$

We shall check that $\lambda > 0$, and this implies, by Gronwall's Lemma,

$$(u(t) - \alpha_-) \leq (u(0) - \alpha_-)e^{-\lambda t} \rightarrow 0 \quad \text{when } t \rightarrow +\infty.$$

It remains to prove that $\lambda = -3\beta^2(1+\alpha_-) + (5\beta^2+4) > 0$ that is $1+\alpha_- < 2a$, i.e. $\sqrt{a^2 - a + \frac{1}{6}} > 1-a$. Either $\beta \leq 2$ and then $a \geq 1$ and we are done, or $\beta > 2$, $a < 1$, and we have to show that $a^2 - a + \frac{1}{6} > (1-a)^2$ i.e. $a > 5/6$ which is true.

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